Divination with Hexagrams as Combinatorial Practice

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A paradigmatic model in mathematics

As a historian of mathematics in China, I am particularly interested in the cultural contexts out of which mathematical procedures grew to solve combinatorial questions: logic, grammar, lottery, games of chance or strategy, and interrupted games all were playgrounds of combinatorial practice to explore experimentally and theoretically the number of possible outcomes, combinations or permutations.
Judging from the few mathematical sources available in China, it seems, that the formation and transformation of the lines in a hexagram were the paradigmatic model on the basis of which number theoretic patterns were to be observed inductively and independently of cosmological considerations. This went as far as considering diagrams with more than six lines, or bringing the broken and unbroken lines

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to such level of abstraction, that an alternative ordering of the sixty-four hexagrams emerged from their purely mathematical interpretation.

In this session, I focused upon a late Qing dynasty text by Wang Lai 汪萊 (1768-1813), *The Mathematical Principles of Sequential Combinations* (*Dijian shuli* 遞兼數理), which intends to make apparent the underlying 'principles' (*li* 理) of two general procedures solving the following two combinatorial questions:

1. when choosing k objects without repetition from a set of n, how many combinations C_k^n are there?
2. what is the sum $S_n = \sum_{k=1}^{n} C_k^n$ of all these combinations C_k^n for $k = 1, 2, \ldots n$?

To justify the procedures solving these questions, Wang relies on diagrams with a separate explanatory discourse using (incomplete) induction. He gives only one example drawn from divination with hexagrams and asking how many transformations of lines are possible, when we start from a certain hexagram (the answer is 63). There is an even earlier manuscript by Chen Houyao 陳厚耀 (1648-1722), the *Meaning of Methods for Alternation and Combination* (*Cuozong fayi* 錯綜法義), that links explicitly combinatorial practices in hexagram divination to mathematics.

Chen's essay deals systematically with problems of permutation and combination in the case of divination with trigrams, the formation of hexagrams or names with several characters, combinations of the ten heavenly stems (*tiangan* 天干) and the twelve earthly branches (*dizhi* 地支) to form the astronomical sexagesimal cycles. Games of chance such as dice throwing and card games equally serve as a model to discuss algorithms for calculating combinations with or without repetition. In the foreword to his treatise, Chen Houyao underlines the originality of his contribution to the mathematical tradition in China but explicitly links the expression *cuozong* in the title of his treatise to the *Book of Change*:

> The *Nine Chapters*[2] have entirely provided all [mathematical] methods, but they lack of any type of method for alternations and combinations. The [Book of] *Change* says: ‘By three, by five, through the transformations; alternating and combining [*cuozong*] the numbers.’[3]
> By ‘alternating and combining’, one forms the numbers themselves from heaven and earth. As for example, by mutually alternating pairs even and odd, one forms the hexagrams, by mutually alternating the stems and branches, one forms the calendar, by mutually alternating the colors,

2 Reference to *The Nine Chapters on Mathematical Procedures* (*Jiu zhang suan shu* 九章算術), the foundational and canonical work of mathematics in ancient China, compiled approximately during the first century A.D.

3 See James Legge, (trans.). *The Yî King*, volume16 of Sacred Books of the East. the Clarendon Press, Oxford, 1882, p. 369-370:

> [The stalks] are manipulated by threes and fives to determine [one] change; they are laid on opposite sides, and placed one up, one down, to make sure of their numbers; and the [three necessary] changes are gone through with in this way, till they form the figures pertaining to heaven or to earth. Their numbers are exactly determined, and the emblems of (all things) under the sky are fixed.

> one forms the brocade, by mutually alternating the five sounds, one forms the melodies. When one pushes this further, from ten to one hundred thousands, there is not one that would not as a consequence of alternation have the charm of the have the charm of the inexhaustible.[4]

The very first problem that Chen states in his text takes the hexagrams as an abstract model for combinatorial considerations. He shows two ways to calculate the possible numbers of combinations in configurations with an abritrary number of lines. Both ways relate constructively to the hexagrams:

1. one can either superpose one line after the other, the number of configurations with n lines then is calculated as 2^n. In the case of the hexagrams, i.e. a configuration made up of six lines, each either broken or unbroken, he underlines that the calculation of all possible combinations (with repetition) can either be obtained by successive multiplication of the two possibilities:

   Number of configurations consisting of 2 lines $= 2 \cdot 2 = 4$

   Number of configurations consisting of 3 lines $= 4 \cdot 2 = 8$

   $\ldots$

   Number of configurations consisting of 6 lines $= 32 \cdot 2 = 64$

4 Chen Houyao 陳厚耀. *Cuozong fayi* 錯綜法義 (The Meaning of Methods of Combination and Alternation). End of 17th cent. Reprint in Guo Shuchun et al. 郭書春 (eds.). *Zhongguo kexue jishu dianji tonghui*. *Shuxue juan* 中國科學技術典籍通彙. 數學卷, 5 vols. Henan jiaoyu chubanshe 河南教育出版社, Zhengzhou, 1993. vol. 4, pp. 685-688, here p. 685.

2. or one can superpose repeatedly, say n times, entire trigrams (of which there are eight). In this case one finds the total number of possibilities by considering 8^n.

Here is the problem-answer-procedure text[5]:

> Let us suppose that the odd line is the *Yang*, and that the even line is the *Yin*. One even or one odd, one superposes until one obtains six lines. How many hexagrams does one obtain? [The answer] says: Sixty-four hexagrams.
> The method says: One even, one odd, by counting this makes two. If one multiplies two by two, one obtains the four diagrams with two lines. If one multiplies again by two, one obtains the eight diagrams of three lines. If one multiplies again by two, one obtains the sixteen diagrams of four lines. If one multiplies again by two, one obtains the thirty-two diagrams of five lines. If one multiplies again by two, one obtains the sixty-four diagrams of six lines. If one superposes up to seven lines or more, one obtains the result equally by successively multiplying by two. Alternatively, one multiplies by itself the eight diagrams of three lines, one obtains the sixty-four diagrams of six lines. It is by multiplying by itself the said number obtained, that one saves half of the multiplications.
> Let us suppose that we have the eight trigrams (*bagua* 八卦) Qian 乾, Dui 兑, Li 離, Zhen 震, Xun

5 Translated from [Chen 1993], op cit, vol. 4, p. 685.

> 巽, Kan 坎, Gen 艮, and Kun 坤. By multiplying and superposing them, how many diagrams should we get? By superposing once more, again, how many diagrams should we get?
> The answer says: When superposing at first, 64 diagrams, when superposing once more, 512 diagrams.
> The explanation says: *Gua* 卦 originally do not have three characters [i.e. three trigrams, thus nine lines]. Now, we wish to explore (*qiong* 窮) the numbers of its superpositions, that is the reason why we repeatedly add on to infer (*tui* 推) them. Each time (*mei yi ci* 每一次) when adding on one character [of three lines], one should also repeatedly multiply (*lei cheng* 累乘) this [the number from the previous configuration] by eight. This gives the result.

Jiao Xun 焦循, in his *Explanation of Addition, Subtraction, multiplication and division* (*Jiajian chengchu shi* 加減乘除釋, 1797), goes even a step further in abstracting from the cosmological implications of the hexagrams, and links the transformations of lines in a hexagram to the Arithmetic Triangle. It first appeared in China in a chapter on algorithms for root extraction, in Yang Hui's 楊輝 *Detailed explanations of The Nine Chapters on mathematical methods* (*Xiangjie jiu zhang suanfa* 詳解九章算法, completed in 1261), but we know, that it must have been circulating a century earlier.
The fact that Jiao's diagram ends with the sixth power of a binomial (see the coefficients 1, 6, 15, 20, 15, 6 and 1, whose sum is 64, in the bottom line of figure 0.1) is explained by Jiao

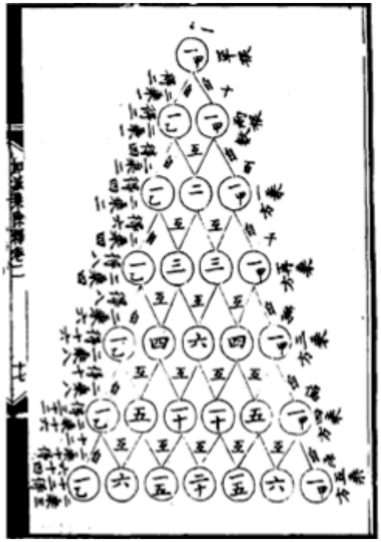


Figure 0.1: Jiao Xun 焦循, *Jiajian chengchu shi* 加減乘除釋 (1797)

Xun as follows:

> This carries the signification of hexagrams which end with 64.[6]

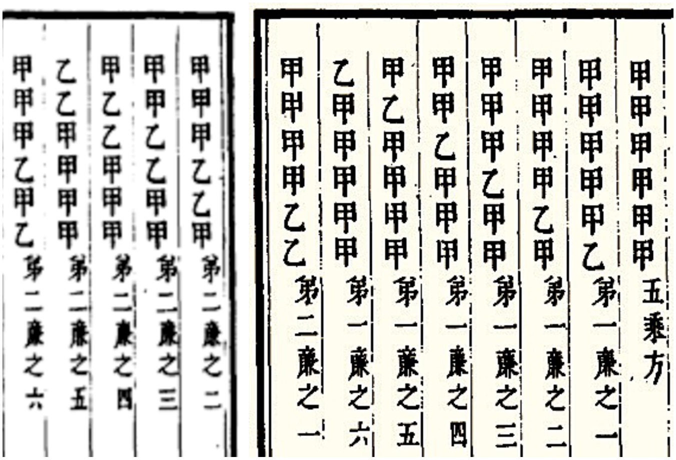
In a manuscript version of the *Explanation of addition, subtraction, multiplication and division*, Jiao Xun uses the arithmetic triangle as a generator of a new order for the hexagrams. He

6 Jiao Xun 焦循. *Litang xuesuan ji* 里堂學算記 (Collection on Mathematical Learning from the Hall of Li). Jiaoshi 焦氏, Jiangdu 江都, 1799, here: *juan* 2 p. 18b.

reads the binomial:

$$(a+b)^6 = 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$$

in terms of possible mutations of the two types of lines of a hexagram. This idea is shown in figure 0.2, where *Jia* and


甲甲甲甲甲甲五乘方
甲甲甲甲甲乙第一廉之一
甲甲甲甲乙甲第一廉之二
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甲乙甲甲甲甲第一廉之五
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甲乙乙甲甲甲第二廉之四
乙乙甲甲甲甲第二廉之五
甲甲甲乙甲乙第二廉之六

Figure 0.2: Jiao Xun 焦循, *Jiajian chengchu shi* 加減乘除釋 (1797)

Yi can be interpreted as the two continuous and interrupted lines respectively. Starting from the hexagram containing six *Jia*-lines (to the very right of figure 0.2, it is also the one possibility, cf. $1a^6$, we have when muting zero lines), there are six possibilities (cf. $6a^5b$) to mute one line. When we mute two lines, there are 15 possibilities (cf. $15a^4b^2$) leading to hexagrams with four *Jia* and two *Yi*-lines, etc.

Wang Lai (1768-1813)

At the center of our reading session was a printed text by Wang Lai 汪萊. Completed in 1799, the text was published in the second half of scroll four of his collected writings,[7] precisely where Horng Wann-Sheng[8] locates the watershed between the Qian-Jia school, 18th-century Chinese mathematics and Wang Lai's studies that led Chinese mathematicians, like Li Shanlan into the 19th century. A closer look at the structure of *The Mathematical Principles of Sequential Combinations* reveals Wang Lai's preoccupation to bring procedure, diagrams and explanations to the forefront, and complement these elements by a paradigmatic numerical example drawn from the realm of divination. The text contains the following elements in sequence:

- A general introduction to the subject, which gives the following elements:
  - (lines 1 to 16) an explanation of what 'configurations of sequential combinations' (*dijian zhi shu* 遞兼之數) generally are;
  - (lines 17 to 33) a method to obtain the 'total number of sequential combinations' (*dijian zhi zongshu* 遞兼之總數): $S_n = \sum_{k=1}^{n} C_k^n$.
  - (lines 34 to 66) a method to obtain the 'partial number of sequential combinations' (*dijian zhi fenshu* 遞兼之分數): C_k^n.

7 *Juan* 4 of *Hengzhai's Mathematics* (*Hengzhai suanxue* 衡齋算學). Reprint see [Guo 1993] op cit. vol. 4, p. 1512-1516.

8 Horng Wann-Sheng. 洪萬生 . Qingdai shuxuejia Wang Lai de lishi dingwei 清代數學家汪萊的歷史定位 (The Place of Wang Lai in the History of Chinese Mathematics). *New History Journal* 新史學, 11(4):1-16, 2000.

- (lines 67 to 87) A 'diagrammatic explanation' (*tujie* 圖解) of the total number of sequential combinations for 10 objects': S_n for $1 \leq n \leq 10$.

- (lines 88 to 93) A 'diagrammatic explanation' (*tujie* 圖解) of the partial number of sequential combinations for 10 objects', the C_k^{10} for $k = 1, \ldots 9$.

- (lines 94 to 134) Five explanations (*jie* 解)[9] concerning the use of the procedures for 'triangular piles' for the 'partial numbers of sequential combinations'.

- (lines 135 to 167) An example: a problem related to divination;

- (lines 168 to 197) A procedure to calculate plane triangular piles:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{1 \cdot 2},$$

  a procedure to calculate 'solid triangular piles':

$$\sum_{k=1}^{n} \frac{k(k+1)}{1 \cdot 2} = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3},$$

  a general method to calculate 'triangular piles', or in modern mathematical terms, the sums of finite arithmetic series of higher orders.

- (lines 198 to end) An example: calculation of the 'fourth-order triangular pile' for $n = 5$.

After an introductory definition of his subject matter, Wang Lai thus illustrates his general method to calculate the total sum

9 The term *jie* is used in the translation of Euclid's *Elements* to designate a first part of the demonstration (*lun*), which is a rewriting of the proposition to be proofed with reference to the particular diagram of that proposition.

of combinations $S_n = \sum_{k=1}^{n} C_k^n$. The indicated algorithm corresponds in modern mathematical terms to a recursive procedure: successively one doubles the `root', i.e. the preceding result, and adds one unity. Given a set of n objects, and starting off with $S_1 = 1$, Wang prescribes $n-1$ iterations of the following operations for $k = 2, \ldots n$:

$$S_k = 2 * S_{k-1} + 1.$$

The corresponding figure (see figure 0.3) depicts for $n = 10$ the $n-1$ iterations of these operations, successively doubling in length and extending by a unitary element a horizontal bar. One thus obtains $S_{10} = 1023$.

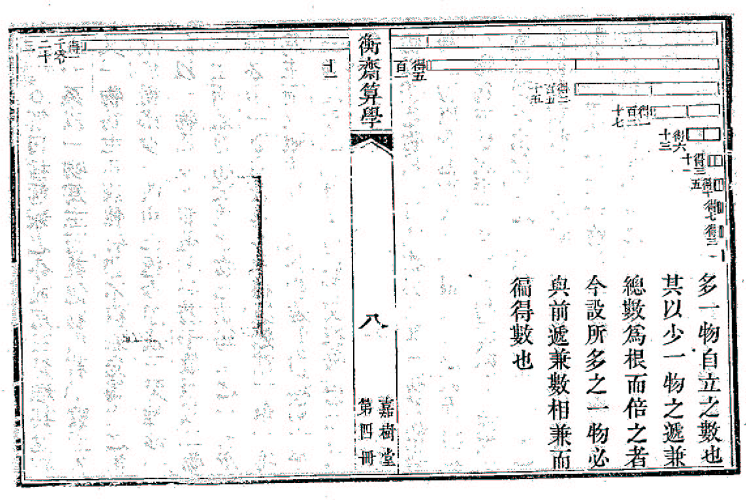

Figure 0.3: Wang Lai (1768-1813), *Mathematical Principles of Sequential Combinations*

Some historians[10] claim that Wang Lai recognized here the remarkable identity:

$$\sum_{i=1}^{n} C_i^n = 2^n - 1.$$

But no explicit mention of the fact that $1023 = 2^{10} - 1$, nor of any kind of generalization can be found in his text.
The following procedure calculates the possible outcomes of drawing i objects out of a set of n objects, which correspond to sums of higher order series. It is an extension from the Yuan dynasty Chinese tradition of considering piles of discrete objects in different geometric shapes as 'figurate numbers'.[11] Zhu Shijie 朱世傑 had already calculated certain of these sums in his *Jade Mirror of Four Elements* (*Siyuan yujian* 四元玉鑑, 1303), but without explicitly referring to problems of combination. For the first time in the transmitted Chinese

10 See for example Li Zhaohua 李兆華. Wang Lai «Dijian shuli», «Sanliang suanjing» lüelun 汪萊《遞兼數理》,《參兩算經》略論 (A short discussion of the «Mathematical Principles of sequential combinations» and the «Mathematical Classic of two and three» by Wang Lai). In Wu Wenjun 吳文俊 (ed.), *Zhongguo shuxueshi lunwenji* 中國數學史論文集 (China Historical Materials of Science and Technology), volume 2, pages 65-78. Shandong jiaoyu chubanshe 山東教育出版社 , Jinan, 1986, or Liu Dun's 劉鈍 introduction to Wang Lai's 汪萊 *Dijian shuli* 遞兼數理 (Mathematical principles of sequential combinations). In: Hengzhai suanxue 衡齋算學 (Hengzhai's Mathematical Learning), volume 4, pages 6b-12b. Jiashutang 嘉樹堂 , China, 1854. Reprint in [Guo 1993] op cit. vol. 4, pp. 1512-1516, here p. 1479.

11 For an extensive discussion of the strands of this tradition, see Andrea Bréard. *Re-Kreation eines mathematischen Konzeptes im chinesischen Diskurs: Reihen vom 1. bis zum 19. Jahrhundert*, volume 42 of Boethius. Steiner Verlag, Stuttgart, 1999.

mathematical tradition, Wang Lai links here combinations to finite sums of arithmetical sequences. He gives drawings for C_i^{10}, illustrating for the example of ten objects the sum of finite series with surfaces and piles of unit pebbles (see figure 0.4 for $i = 1, \ldots 5$ from right to left). He also remarks the symmetry $C_i^{10} = C_{10-i}^{10}$, which explains why he does not illustrate the cases C_i^{10} for $i = 6, \ldots 10$, but only the cases, where sequentially one, two, three, four or five objects are drawn from a set of ten objects.

When Wang Lai calculates the total number of pebbles lined or piled up in triangular or pyramidal shape, the so called 'triangular piles' (*sanjiao dui* 三角堆), he uses the following procedures for finding the sums of finite arithmetical series of higher order, which were (except for C_5^{10}) known to Zhu Shijie :

$$C_1^{10} = C_9^{10} = 1 + 1 + \ldots + 1 = 10$$

$$C_2^{10} = C_8^{10} = 1 + 2 + 3 + \ldots + 9 = \sum_{k=1}^{9} k = \frac{9 \cdot 10}{2} = 45$$

$$C_3^{10} = C_7^{10} = 1 + 3 + 6 + 10 + \ldots + 36 = \sum_{k=1}^{8} \frac{k(k+1)}{2} =$$

$$= \frac{8 \cdot 9 \cdot 10}{2 \cdot 3} = 120$$

$$C_4^{10} = C_6^{10} = 1 + 4 + 10 + 20 + \ldots + 84 = \sum_{k=1}^{7} \frac{k(k+1)(k+2)}{6} =$$

$$= \frac{7 \cdot 8 \cdot 9 \cdot 10}{2 \cdot 3 \cdot 4} = 210$$

$$C_5^{10} = 1 + 5 + 15 + 35 + 70 + 126 = \sum_{k=1}^{6} \frac{k(k+1)(k+2)(k+3)}{24} =$$

$$= \frac{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{5 \cdot 4 \cdot 3 \cdot 2} = 252$$

The illustrations of C_i^{10} in figure 0.4 (where $i = 1, \ldots 5$ from top right to left) suggest the patterns of formation of every term of the above series. The series $C_2^{10} = 1 + 2 + 3 + \ldots + 9$ thus becomes a triangle in which pebbles are piled up in rows with 1 to 9 pebbles in each successive row. The next sum is then a regular pyramid, where each layer is composed of one of such triangles, each having (from top to bottom) 1, 3, 6, …and 36 elements. For C_4^{10}, for example, Wang shows seven triangular pyramids with one to seven layers:

$$C_4^{10} = 1+(1+3)+(1+3+6)+\ldots+(1+3+6+10+15+21+28)$$

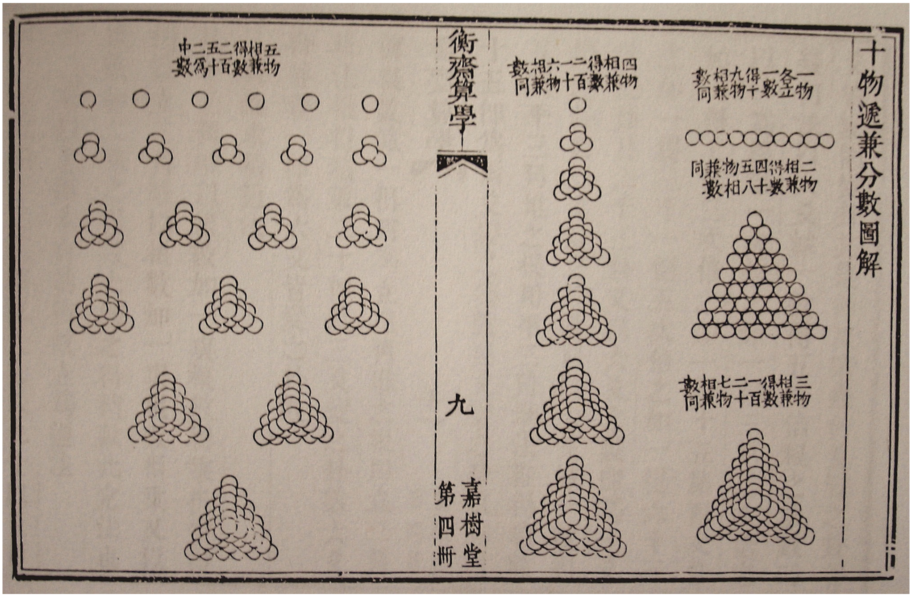

Figure 0.4: 'Diagrammatic explanation of the partial number of sequential combinations for 10 objects' in Wang Lai (1768-1813), *Mathematical Principles of Sequential Combinations*

Finally, C_5^{10} is represented as 21 pyramids which can be grouped in two ways. A horizontal reading of the drawings in the left half of figure 0.4 gives the terms:

$$C_5^{10} = 6 \cdot 1 + 5 \cdot 4 + 4 \cdot 10 + 3 \cdot 20 + 2 \cdot 35 + 1 \cdot 56$$

whereas a diagonal reading from left to right produces different terms for the same sum:

$$C_5^{10} = 1 + (1+4) + (1+4+10) + (1+4+10+20)+$$
$$+(1+4+10+20+35) + (1+4+10+20+35+56).$$

As an application of Wang Lai's procedure to calculate S_n, only one related mathematical problem is stated in his text. It stems from the earliest witness of combinatorial practices, divination with hexagrams. In Wang's example, a shaman performing yarrow stalks divination (*shigua* 筮卦) produces a hexagram, a configuration made up of six lines (*liu yao* 六爻). Wang is interested in the total number of possible transformations of the one to six lines, that one can produce with a hexagram. Mathematically, this corresponds to finding the sum of $C_1^6 + C_2^6 + C_3^6 + C_4^6 + C_5^6 + C_6^6$. He calculates his result, not by summing up the C_k^6, but by using the first recursive method introduced in the beginning of his essay to calculate the ‘total number of sequential combinations’. He proceeds by doubling successively the minimum number of lines in such a configuration and then adding one. Wang Lai remarks that five (i.e. the maximum number of lines that one can obtain minus one) iterations give the total number of possible configurations. In five steps he calculates the result:

$$2 \cdot 1 + 1 = 3$$
$$2 \cdot 3 + 1 = 7$$
$$2 \cdot 7 + 1 = 15$$
$$2 \cdot 15 + 1 = 31$$
$$2 \cdot 31 + 1 = 63$$

In a second step, Wang Lai calculates the possibilities to mute one to six lines of a given hexagram, which mathematically corresponds to the 'partial' C_k^6. He does calculate these by using the procedures for 'triangular piles'. Again, as Wang Lai recognized the symmetry $C_m^n = C_{n-m}^n$ he does not have to go beyond the calculation of C_3^6:

$$C_1^6 = C_5^6 = 6$$

$$C_2^6 = C_4^6 = 15 = \frac{5 \cdot (5+1)}{2} = \sum_{k=1}^{5} k$$

$$C_3^6 = 20 = \frac{4 \cdot (4+1) \cdot (4+2)}{6} = \sum_{k=1}^{4} \frac{k(k+1)}{2} = \sum_{k=1}^{4} \sum_{i=1}^{k} i$$

$$C_6^6 = 1$$

$$\sum_{k=1}^{6} C_k^6 = 63$$

It is somewhat surprising, that Wang Lai does not bring his calculations in connection with the arithmetic triangle, as did Jiao Xun, his contemporary and close friend. Its seventh line would contain precisely the C_n^6 for $n = 0, \ldots, 6$ (i.e. the numbers 1, 6, 15, 20, 15, 6 and 1), and their sum equals 2^6. But Wang does not refer here to the corresponding values in the triangle. And again, no explicit mention of the fact that $63 = 2^6 - 1$ nor of any kind of generalization to $S_n = 2^n - 1$ can be found in his text.

What Wang gives in the end, are the procedures for 'plane triangular piles' (the sum of the natural numbers), for 'solid triangular piles' (the sum of the sums of natural numbers) and the general procedure for the sum of higher order 'triangular piles' (i.e. finite arithmetic series):

$$\frac{(n+1)\cdot(n+2)\cdot\ldots(n+m)}{1\cdot 2\cdot\ldots m} = C_m^{n+m}.$$

Finally, as an example for his general procedure, Wang explicitly formulates the procedure and performs numerical calculations to determine the sum of the so called 'fourth-order triangular pile' (*si cheng sanjiao dui* 四乘三角堆)[12] with five as the particular 'base number' (*genshu* 根數):

$$C_4^{4+5} = C_5^9 = \frac{5\cdot(5+1)\cdot(5+2)\cdot(5+3)\cdot(5+4)}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 126$$

Altogether, Wang Lai did constitute a point of synthesis of two independent mainstreams interested in 'combinatorial' results, the 'figurate numbers' stemming from Zhu Shijie's mathematical researches on triangular piles, and the combinatorial practices, going back to the divinatory devices of the *Yijing*. It seems, that knowledge and practices related to the sixty-four hexagrams, in particular considerations of mutations of lines, were the paradigmatic model for "early" combinatorial algorithms in China.

Complete translation of Wang Lai's text

"Procedures (*shu* 數) of sequential combinations had not been discovered in ancient times. Now that I have decided to inves-

12 Literally the Chinese term here rendered as 'order' can be translated as 'multiplication', since, as Chen points out himself, this number corresponds to the number of multiplications to be performed to calculate each, the dividend and the divisor.

tigate them, it is thus appropriate to explain the object of inquiry first. Let us suppose one has all kinds of objects. Starting off from one object of which each establishes one configuration (*shu* 數), and going up to all the objects taken together, they form altogether one configuration. In between lie sequentially: two objects connected to each other forming one configuration, we shall discuss how many configurations this can make through exchanging and permuting (*jiao cuo* 交錯); three objects connected to each other forming one configuration, we shall discuss how many configurations this can make through exchanging and permuting; four objects, five objects, up to arbitrarily many objects, not one doesn't entirely follow those which is the so called procedures of sequential combinations.

When we want to seek/determine how much makes the total of [these] numbers [the number of ‘sequential combinations’, or in mathematical terms: the sum of the C_k^n], and how much makes each partial number [the C_k^n themselves], one distinguishes two methods.

[The first] method: One takes the supposed number of objects [n] and subtracts one unit/the number one [$= n - 1$]. This gives the number of times one will have to ‘double the base’. Thus, one takes one as the [first] base [C_1^1], doubles it and adds one. We obtain three as the [result] of the first [iteration] [$C_1^2 + C_2^2$]. Again, one doubles this and adds one to obtain seven as the [result] of the second [iteration] [$C_1^3 + C_2^3 + C_3^3$]. In this way, one successively doubles and successively adds one until one arrives at the corresponding number of times [i.e. $n - 1$ iterations], where one stops. What one obtains in the end is the total number of mutual combinations (*xiang jian* 相兼).

[The second] method: Again, one takes the supposed number of objects [n] - which is in fact the number when each [object] constitutes individually a configuration [C_1^n][13] - and subtracts one. This makes the base of a triangular pile.[14] Now, when by taking this base number one seeks/determines the resulting plane triangular pile, it makes the number of mutual combinations of two objects [C_2^n]. Again, subtracting one unit/the number one, when one seeks/determines the resulting solid triangular pile, it makes the number of mutual combinations of three objects [C_3^n]. Again, subtracting one unit/the number one, when one seeks/determines the resulting four-dimensional triangular pile,[15] it makes the number of mutual

13 The idea of Wang Lai here to link the number of objects to the number of possible combinations one out of n, corresponds mathematically to the equality $C_1^n = n$. In Wang's second diagram, with $n = 10$, the number C_{10}^1 is represented on the upper right side, by an alinement of ten unitary pebbles.

14 This is the number of units at the base of a triangle or a pyramid constituted of unitary elements. With $b = n - 1$, a 'triangular pile' in the plain, for example, would be constituted of b elements at the base of the triangle, of $b - 1$ elements in the row above the base, of $b - 2$ elements placed above, etc. up to a single element at the tip of the triangle. Altogether this makes $b + (b-1) + \ldots + 2 + 1$ elements, which is equal to $\frac{n(n-1)}{2}$ elements, or to C_2^n as Wang Lai indicates in the following sentence.

15 Translated into anachronistic modern mathematical language, this corresponds to the passage from the sum:

$$\sum_{k=1}^{n-2} \frac{k(k+1)}{1 \cdot 2}$$

to the sum of the arithmetic series of one order higher:

$$\sum_{k=1}^{n-3} \frac{k(k+1)(k+2)}{1 \cdot 2 \cdot 3}.$$

combinations of four objects [C_4^n].[16]

In this way, one successively subtracts from/diminishes the base number and successively augments the number of dimensions to seek/determine the resulting various numbers of mutual combinations, until one arrives at the median number, where one stops. Beyond the central/median number, it [the procedure] is the same as before, one does not need to redo the calculations backwards. The central/median number is positioned in the middle, determined by the 'remaining number' [of steps to perform] when one has taken away/subtracted from the originally supposed number of objects [n] up to the constellation with the most [elements]. When the 'remaining number' is odd, then there is a single centre/median. When it is even, then there are two centres/medians. In case there are two centers, their numbers of mutual combinations are also equal.[17] Such are the partial numbers of sequential combinations (*dijian zhi fenshu* 遞兼之分數). Now, we give below explanation with diagrams, using ten objects. When pushing this further to hundred, thousand, ten thousand or one hundred million, there will be none, that does not conform to the same principle!"

16 In Wang's diagram, the number C_4^{10} is represented by a set of seven triangular pyramids, of which the smallest has one pebble at its base, and the biggest one seven [$10 - 3 = 7$] pebbles at the side of the base triangle.

17 If, for example, one calculates the C_k^{10}, one has two 'centers/medians' with the same value: C_5^{10} and C_6^{10}, and there is an even number of C_k^{10} to be determined by symmetry, i.e. the four values $C_7^{10} = C_4^{10}$, $C_8^{10} = C_3^{10}$, $C_9^{10} = C_2^{10}$, $C_{10}^{10} = C_1^{10}$.

Diagram and Explanation of the total number of sequential combinations of ten objects.

gives 1023	9th
gives 511	8th
gives 255	7th
gives 127	6th
gives 63	5th
gives 31	4th
gives 15	3rd
gives 7	2nd
gives 3	1st
	double & add
1	base

The explanation says: 'Adding one unit', is the configuration established individually by a supposed extra object. The total number of sequential combinations of the objects diminished by one makes the base [S_{n-1}]. 'Doubling it': one has to mutually combine the configuration established individually by the supposed extra object with the previous number of sequentially combined [objects] in order to obtain all the configurations.

Diagram and Explanation of the partial number of sequential combinations of ten objects.

[diagrams with piles of unitary spheres]…

The explanation says: When deducing the partial number of sequential combinations we use triangular piles. There are five explanations to this.

The first one is, that by taking single objects as the dominant [element],[18] and by combining the other objects, one obtains

18 *Zhu* 主, lit. host (who invites). In cosmology: to exerce a domination in

their amount. If then, by taking one more object as the dominant [element], and by combining the other objects, one does not have to combine again the object which was previously been taken as the dominant [element]. This is why that which one obtains has to be smaller by one number. From there/by this, sequentially subtract and subsequently construct a triangular form.

Another one [another explanation] is, that by taking one object as the dominant [element], and by combining the other objects, one obtains their amount. If then, by taking two objects as the dominant [elements], and by combining the other objects, the object that has been subjected to combination has already been deduced as one dominant [element]. This is why that which one obtains has to be smaller by one number. From there/by this, sequentially subtract, this is why the base number is sequentially reduced by one.

Another one [another explanation] is, that by taking one object as the dominant [element], and by combining the other objects, one constructs one/the first base. Each object sequentially subtracted, this constructs a plane triangular pile. If then, by taking two objects as the dominant [element], then the one object and the other object together make a combination of two different objects, and construct/form a base. That object and another object again together make two objects. By combining the other objects again one constructs/forms a base. From there/by this, sequentially subtract and proceed to subsequently erect a solid triangular pile. From there/by this

the cycle of the five agents (*wu xing* 五行). In chin. pharm.: the dominant ingredient in the composition of a prescription. In divin.: being an indicator of; prognostic.

sequentially proceed, this is why the number by which is multiplied sequentially augments by one.

Another one [another explanation] concerns what comes before and after the median number. The number of what has been combined before it, and the number of what has not yet been combined after it are equal. This is why the obtained numbers are equal.

Another one [another explanation] is, that the median number is in the middle of each individual object establishing one number [C_1^n] and each individual object not yet being combined [C_{n-1}^n]. This is the reason why we do not consider the one position where all objects are taken together [$C_n^n = 1$].

An example:

Let us suppose that a shaman is divining/determining a hexagram. Each hexagram has six lines. From muting one line up to all the mutations of six lines, one asks in total, how many mutations of hexagrams there are?[19] and how many mutations there are for the configurations of all the different numbers of lines?[20] The method is to take the six lines, subtract the number one. This gives five as the number of times one will have to 'double the base'. Thus, one takes one as the [first] base. One doubles for the first time and adds one. One obtains three. One doubles this for the second time and adds one. One obtains seven. One doubles this for the third time and adds one. One obtains fifteen. One doubles this for the fourth

19 Wang is interested in the total number of possible transformations of one to six lines, which corresponds to finding the sum of $C_1^6 + C_2^6 + C_3^6 + C_4^6 + C_5^6 + C_6^6$, what he called in the first part of his text the 'total number of sequential combinations', here for $n = 6$.

20 Here, Wang asks for the 'partial numbers of sequential combinations' for $n = 6$: C_1^6, C_2^6, C_3^6, C_4^6, C_5^6 and C_6^6.

time and adds one. One obtains thirty-one. One doubles this for the fifth time and adds one. One obtains sixty-three. The total number of mutations of configurations is sixty-three configurations.

Furthermore, one takes the number of six lines, it is the number of hexagrams with mutations of one line, which is equal to the number of configurations with mutations of five lines.[21] From six lines one subtracts the number one, and obtains five as the basis of a plane triangular pile. One uses the method for plane triangular piles to calculate/deduce the result of fifteen as the product-/surface-number. This is the number of configurations with mutations of two lines, which is equal to the number of configurations with mutations of four lines.[22] From the previous base-number one subtracts one, and obtains four as the basis of a solid triangular pile. One uses the method for solid triangular piles to calculate/deduce the result of twenty as the product-/surface-number. This is the number of configurations with mutations of three lines.[23] Six lines combined together give one,[24] this is the number of configurations with mutations of all six lines.

A general method for determining the accumulation/product/sum of triangular piles.

In general, for a plane triangular pile, one multiplies the base number augmented by one with the base number and halves this. One obtains the accumulation number. For the solid triangular pile, one multiplies the base number augmented by one with the base number; Furthermore one multiplies this with

21 $C_1^6 = C_5^6 = 6.$

22 $C_2^6 = C_4^6 = 15 = \frac{5\cdot(5+1)}{2} = \sum_{k=1}^{5} k.$

23 $C_3^6 = 20 = \frac{4\cdot(4+1)\cdot(4+2)}{6} = \sum_{k=1}^{4} \frac{k(k+1)}{2} = \sum_{k=1}^{4} \sum_{i=1}^{k} i.$

24 $C_6^6 = 1.$

the base number augmented by two. What one obtains is divided by six. One obtains the accumulation number. This is a determined method/a fixed law. On the other hand, from four dimensional [piles] upwards, we do not yet have their procedures. This is why I establish a general method (*tongfa* 通法).

The method [is]: take the base number. Use one, two, three, four, five, six, seven, eight, nine, ten, up to hundreds, thousands, ten thousands, hundred thousands. In respective sequence add to all number seperately up to the ‘multiplication number’ (*cheng shu* 乘數), where one stops. This makes the cumulative multiplication model/pattern (*lei cheng fa* 累乘法). Then put down the base number and cumulatively multiply it with the cumulative multiplication model/pattern (*lei cheng fa* 累乘法). The obtained number makes the dividend. Furthermore put down one as the divisor. First, use one, two, three, four, five, six, seven, eight, nine, ten, up to hundreds, thousands, ten thousands, hundred thousands. In respective sequence multiply all numbers cumulatively. This makes the divisor model/pattern (*chu fa* 除法) of the associated multiplicative triangular pile. With the divisor model/pattern of the determined 'multiplication number' (*cheng shu* 乘數) one divides the previously [found] dividend. One obtains the accumulation number.

An example: Let us suppose that the base number is five, one wants to find the ‘four-multiplicative triangular pile’.[25] Taking five and adding one makes six, adding two makes seven, adding three makes eight, adding four makes

25 I.e. the five-dimensional triangular pile, or: the sum of a series with terms in arithmetic progression of order 5: $C_5^{n+4} = \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{i(i+1)(i+2)}{6} = \frac{n(n+1)(n+2)(n+3)(n+4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$. Here: $C_5^9 = \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 126$.

nine. Since what one wants to find is the ‘four-multiplicative [number]’, we stop at addition of four. What is calculated is six, seven, eight, nine. One has obtained the ‘four cumulative multiplication model/pattern’ (*lei cheng fa* 累乘法). Then, put down the base number five, and cumulatively multiply it. At the first step (*ci* 次), use six to multiply, one obtains thirty. At the second step, use seven to multiply, one obtains 210. At the third step, use eight to multiply, one obtains 1680. At the fourth step, use nine to multiply, one obtains 15120. This makes the dividend. Furthermore, put down one. At the first step, use two to multiply, one obtains two. At the second step, use three to multiply, one obtains six. At the third step, use four to multiply, one obtains twenty-four. At the fourth step, use five to multiply, one obtains 120. Since what one wants to find is the ‘four-multiplicative [number]’, this corresponds to what has been obtained as the divisor model/pattern (*chu fa* 除法) at this fourth step. With this divisor model/pattern 120 divide the previous dividend. One obtains 126 as the accumulation/product/sum number.[26]

26 Translated from [Wang 1854] op cit. vol. 4, p. 7a-12b (p. 1512-1515).